7. Paths and Curves, Cylindrical and Spherical Coordinate Systems

In this lecture, we will discuss

- Paths and Curves $\mathbb{R}^2$ and $\mathbb{R}^3$
- Cylindrical Coordinate Systems
- Spherical Coordinate Systems

Paths and Curves $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition Path and Curve
A path in $\mathbb{R}^3$ (or $\mathbb{R}^2$) is a function $\mathbf{c}:[a,b]\to\mathbb{R}^3$ (or $\mathbb{R}^2$), whose domain is a subset $[a,b]\subseteq\mathbb{R}$. The image of $\mathbf{c}$ is called a curve in $\mathbb{R}^3$ (or $\mathbb{R}^2$). The function $\mathbf{c}$ is also known as a parametrization (or parametric representation or parametric equation) of the curve.

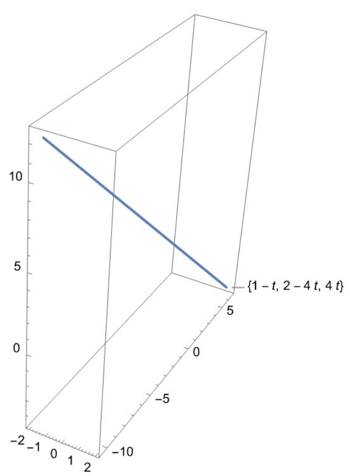
Definition Orientation
Let $\mathbf{c}(t):[a,b]\to\mathbb{R}^3$ (or $\mathbb{R}^2$) be a path. The point $\mathbf{c}(a)$ is called the *initial point*, and we call $c(b)$ the *terminal point* of c. The initial and the terminal points are called the *endpoints* of $\mathbf{c}$. The direction corresponding to increasing values of t gives the positive orientation, whereas the *opposite direction* defines the *negative orientation* of $\mathbf{c}$.

- **Parametric Representation of a Line and a Line Segment**

  Recall in **Lecture 1 Exercise 5**, the parametric equations of the line ℓ in $\mathbb{R}^3$ that contains a point $A=(1,2,0)$ and with direction of a vector $\mathbf{v}=(-1,-4,4)$ is

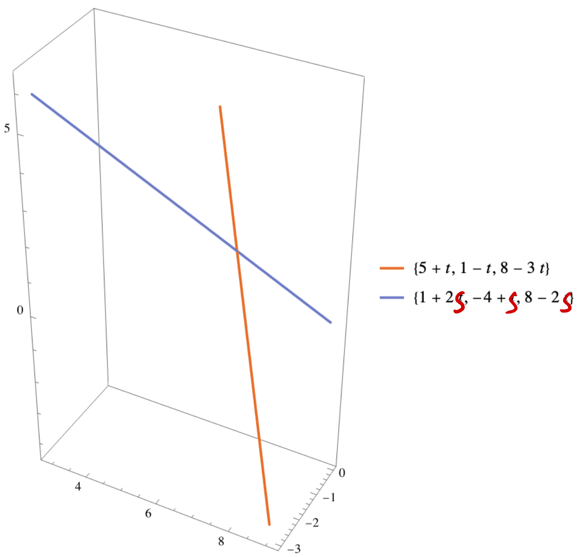
$$\mathbf{r}(t)=\mathbf{a}+t\mathbf{v}=(1-t,2-4t,4t),\quad t\in\mathbb{R}.$$

  Taking $t\in[-1,3]$, we get a line segment.



Example 1. (Related to WebWork Q4)

Find the point of intersection of the two lines L_1 and L_2, where $L_1 = (5+t, 1-t, 8-3t)$ and $L_2 = (1+2s, -4+s, 8-2s)$.



ANS. To find the point of intersection of L_1 and L_2, we must find a point P that lies on both lines. This means if the Line intersect, we must have

$$\begin{cases} 5+t = 1+2s \Rightarrow 4 = 2s-t \quad ① \\ 1-t = -4+s \Rightarrow 5 = s+t \quad ② \\ 8-3t = 8-2s \Rightarrow 0 = 3t-2s \end{cases}$$

$① + ② \Rightarrow 9 = 3s \Rightarrow s = 3$

Plug $s=3$ into ②, we get $t=2$.

To find the point of intersection P, we just substitute $\begin{cases} s=3 \\ t=2 \end{cases}$ into either L_1 or L_2

For L_1, we get

$$L_1 = (5+t, 1-t, 8-3t) = (7, -1, 2)$$

To double check the answer, we plug $s=3$ int

$L_2 = (1+2s, -4+s, 8-2s) = (7, -1, 2)$ as expected.

- **Parameterization of a Circle and an Ellipse**

$x^2 + y^2 = r^2\cos^2 t + r^2 \sin^2 t$
$= r^2$

  - The curve represented parametrically as

$$\mathbf{c}(t) = (r \overset{x}{\cos t}, r \overset{y}{\sin t}), \quad t \in [0, 2\pi]$$

(where $r > 0$) is the circle in $\mathbb{R}^2$ of radius r centered at the origin.

Generally,

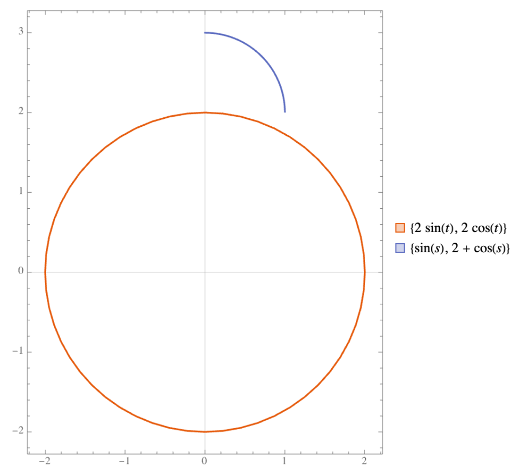
$$\mathbf{c}(t) = (x_0 + r\cos t, y_0 + r\sin t), \quad t \in [0, 2\pi]$$

is a parametric equations of circle of radius r centered at $C = (x_0, y_0)$.

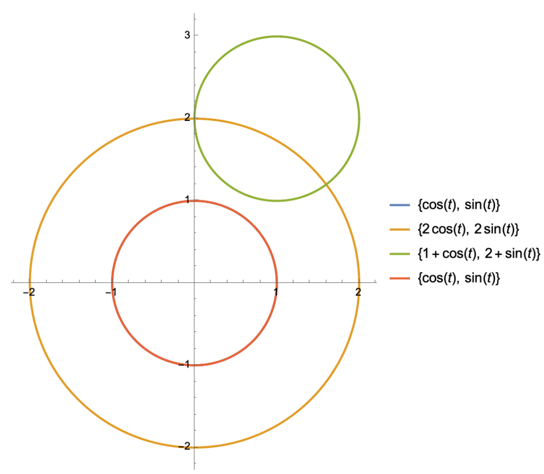
Below is the graph of two parametric equations.

The orange one is obtained by $(2\cos t, 2\sin t), \quad t \in [0, 2\pi]$.

The blue one is by $(\sin s, 2 + \cos s), \quad s \in [0, \pi/2]$. Notice that it is an arc of the circle centered at $(0, 2)$ with radius 1 with an angle range from 0 to $\pi/2$.

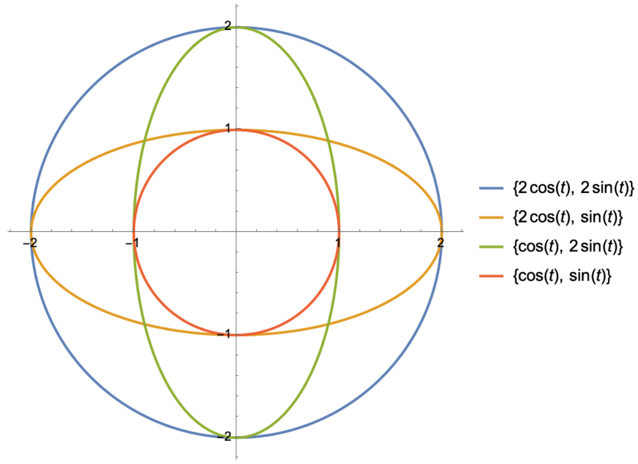


The following graph is an example of serveral circles with their correponding parametric equations.



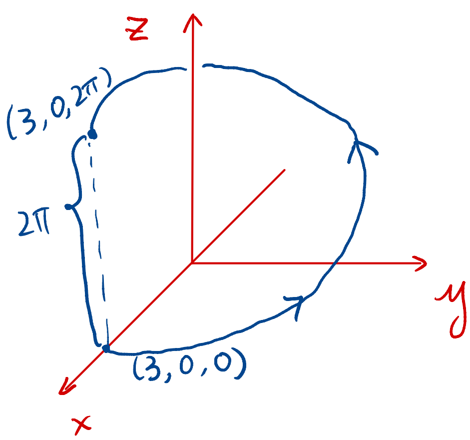
- The ellipse $x^2/a^2 + y^2/b^2 = 1$ (with semiaxes $a, b > 0$) can be parametrized as $c(t) = (a\cos t, b\sin t), t \in [0, 2\pi]$.

  The following graph is an example of taking different values of a and b.



Example 2. (Related to WebWork Q2)

Sketch the curve $\mathbf{c}(t) = (3\cos t, 3\sin t, t), t \in [0, 2\pi]$.



ANS: - $x = 3\cos t$, $y = 3\sin t$ implies $x^2 + y^2 = 9$, therefore the curve lies on the surface of the cylinder of radius 3.

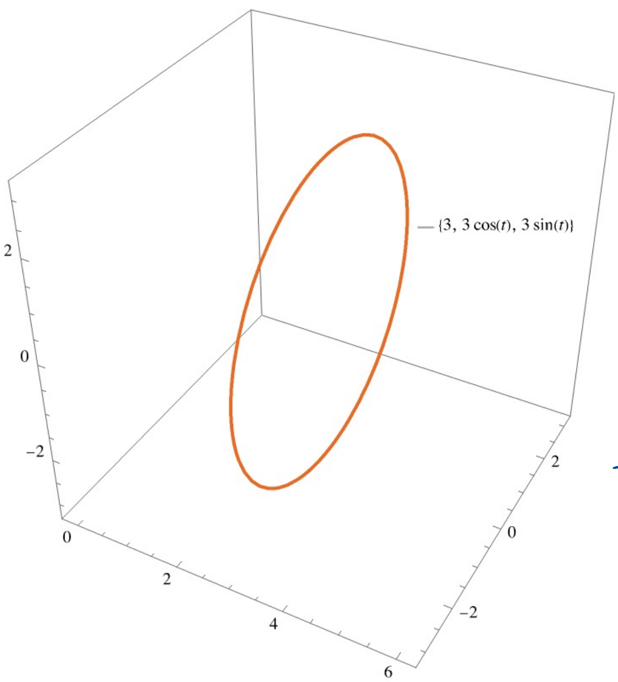
- Its projection onto the xy-plane (take $z = 0$) is the circle of radius 3 oriented counterclockwise.

- As t increases, z coordinate will increase from 0 to 2π. The initial point is $(3, 0, 0)$ and the terminal point is $(3, 0, 2\pi)$.

Example 3. (Related to WebWork Q3)

The function $\mathbf{r}(t)$ traces a circle. Determine the radius, center, and plane containing the circle

$$\mathbf{r}(t) = 3\mathbf{i} + (3\cos(t))\mathbf{j} + (3\sin(t))\mathbf{k}$$



ANS: We have

$x(t) = 3$, $y(t) = 3\cos t$, $z(t) = 3\sin t$.

Thus $y^2 + z^2 = 3^2\cos^2 t + 3^2\sin^2 t$

$= 3^2$

This is the equation of a circle in the verticle plane $x = 3$.

The circle is centered at $(3, 0, 0)$ and the radius is 3.

Cylindrical Coordinate Systems

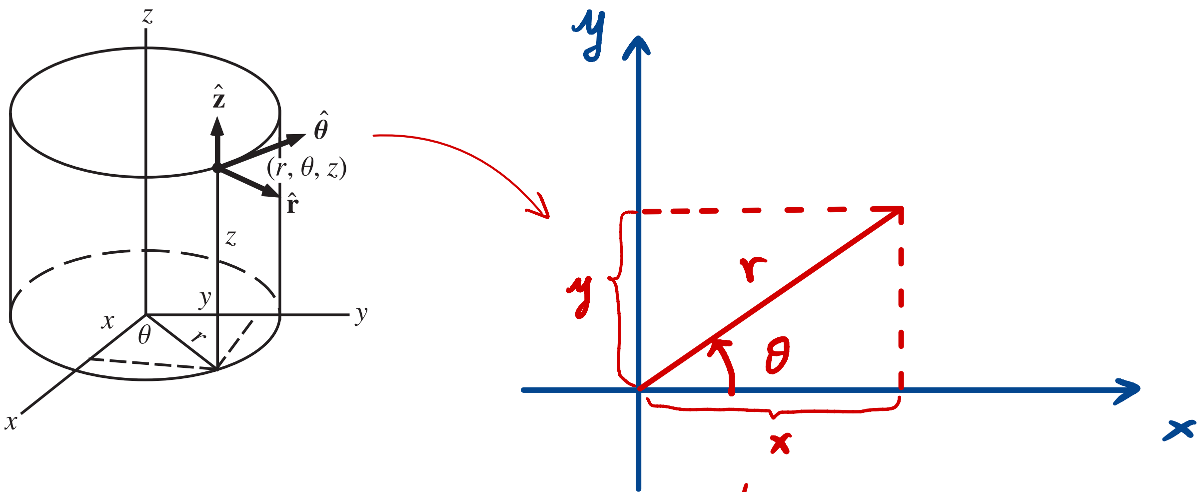
There are many ways of representing points in $\mathbb{R}^3$ other than using the Cartesian coordinates x, y, and z. Two commonly used sets of coordinates (e.g., in integration) are cylindrical and spherical coordinates.

Definition Cylindrical Coordinates r, θ, z

The cylindrical coordinates r, θ, z are related to Cartesian coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

where $0 \le \theta < 2\pi$, and $r \ge 0$.



Note $\tan\theta = \frac{y}{x}$

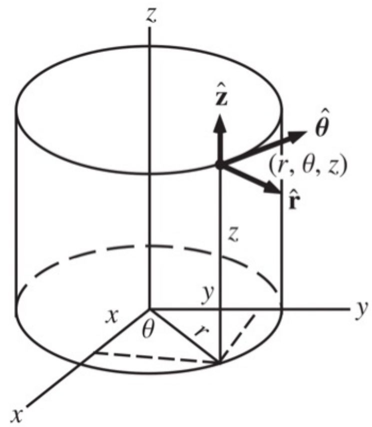
As $-\frac{\pi}{2} < \arctan\theta < \frac{\pi}{2}$, but we need $0 \le \theta < 2\pi$, thus we have the formula computing θ in the table below.

From Cylindrical to Cartesian	From Cartesian to Cylindrical
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$r = \sqrt{x^2 + y^2}$ $\theta = \begin{cases} \arctan(y/x) & \text{if } x > 0 \text{ and } y \ge 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \\ \arctan(y/x) + 2\pi & \text{if } x > 0 \text{ and } y < 0 \end{cases}$ $z = z$

Example 4. (Related to WebWork Q7)

Convert the following point from Cartesian (rectangular) to cylindrical coordinates:

$$(x, y, z) = (\sqrt{3}, -1, -5).$$



ANS: We are give

$(x, y, z) = (\sqrt{3}, -1, -5)$

$r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= 2$

We know

$\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}}$

$\arctan\theta = -\frac{\pi}{6}$

Since $(x, y) = (\sqrt{3}, -1)$ lies on the fourth quadrant, $\frac{3\pi}{2} \leq \theta < 2\pi$. Thus

$\theta = 2\pi + \arctan\theta = \frac{11\pi}{6}$.

$z = -5$

Thus

$(r, \theta, z) = (2, \frac{11\pi}{6}, -5)$

Exercise 5. (Related to WebWork Q10)

Convert the following point from cylindrical to Cartesian (rectangular) coordinates:

$$(r, \theta, z) = (3, \pi, -9.5).$$

Solution. We have $r = 3, \theta = \pi$, and $z = -9.5$. Thus,

$$\begin{aligned} x &= r \cdot \cos(\theta) = 3 \cdot -1 = -3 \\ y &= r \cdot \sin(\theta) = 3 \cdot 0 = 0 \\ z &= -9.5 \end{aligned}$$

Spherical Coordinate Systems

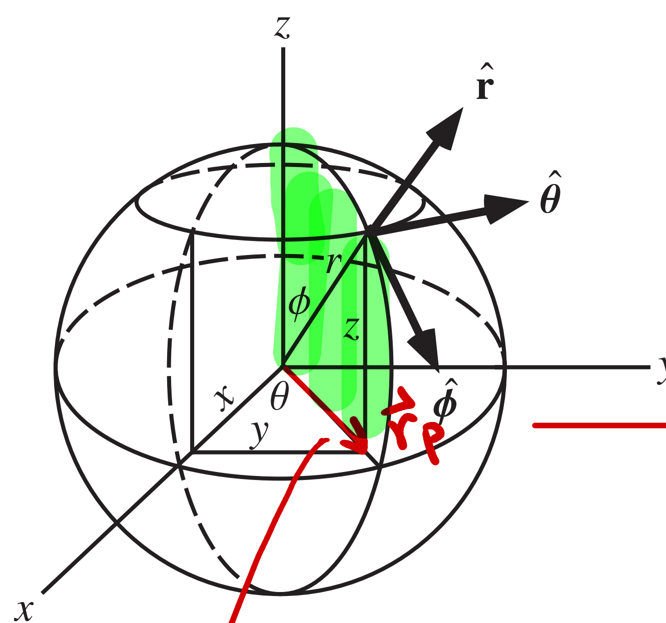
Definition Spherical Coordinates ρ, θ, ϕ

The point $(x, y, z) \in \mathbb{R}^3$ is represented in *spherical coordinates* using the following data:

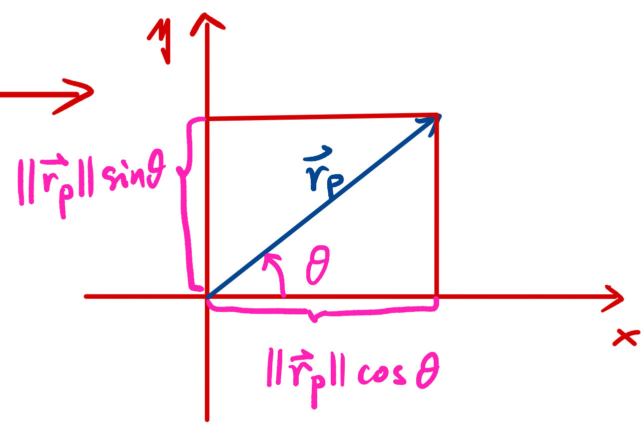
(a) Distance $\rho = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2} \geq 0$ from the origin.

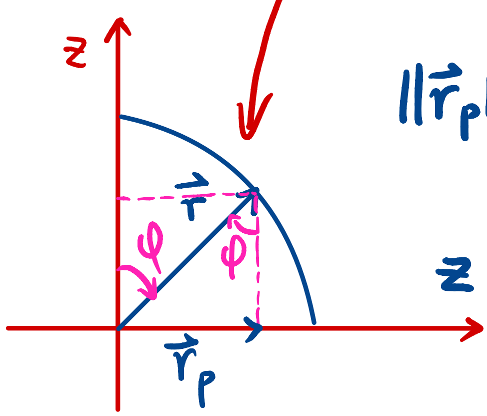
(b) Angle $\theta (0 \leq \theta < 2\pi)$ in the xy-plane (measured counterclockwise) between the x-axis and the projection $\mathbf{r}_p$ of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ onto the xy-plane.

(c) Angle ϕ (in the plane containing the z-axis and the position vector $\mathbf{r}$, measured from the positive direction of the z-axis), $0 \leq \phi \leq \pi$. If a point lies on the z-axis, then $\phi = 0$ if $z \geq 0$ and $\phi = \pi$ if $z < 0$.



projection onto xy-plane





$$\|\vec{r}_p\| = \|\vec{r}\| \cdot \sin\varphi = \rho \sin\varphi$$

$$z = \|\vec{r}\| \cos\varphi = \rho \cos\varphi.$$

$$x = \|\vec{r}_p\| \cdot \cos\theta = \rho \sin\varphi \cos\theta$$

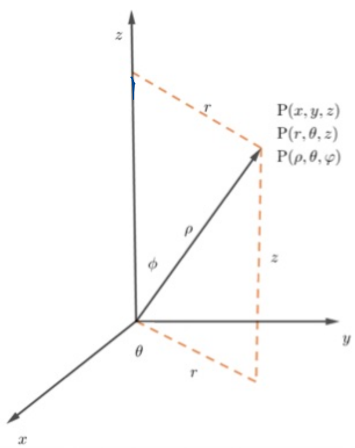
$$y = \|\vec{r}_p\| \cdot \sin\theta = \rho \sin\varphi \sin\theta$$

From Spherical to Cartesian	From Cartesian to Spherical
$x = \|\mathbf{r}_p\| \cos\theta = \rho \sin\phi \cos\theta$ $y = \|\mathbf{r}_p\| \sin\theta = \rho \sin\phi \sin\theta$ $z = \rho \cos\phi.$	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\theta = \begin{cases} \arctan(y/x) & \text{if } x > 0 \text{ and } y \geq 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \\ \arctan(y/x) + 2\pi & \text{if } x > 0 \text{ and } y < 0 \end{cases}$ $\phi = \arccos(z/\rho).$

Example 6. (Related to WebWork Q8)

Convert the following point from Cartesian (rectangular) to spherical coordinates:

$$(x, y, z) = \left(\frac{-5\sqrt{6}}{4}, \frac{-5\sqrt{2}}{4}, \frac{5\sqrt{2}}{2}\right).$$



Solution: We suggest to draw down the diagram on the left for this type of questions so that you can derive the formula yourself

To find $P(\rho, \theta, \varphi)$, we have

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\frac{25 \times 6}{16} + \frac{25 \times 2}{16} + \frac{25 \times 2}{4}}$$

$$= \sqrt{\frac{25(6+2+8)}{16}} = 5$$

$$\tan\theta = \frac{y}{x} = \sqrt{\frac{2}{6}} = \sqrt{\frac{1}{3}}$$

As $y < 0$, $x < 0$, θ lies in the 3rd quadrant, thus $\pi < \theta < \frac{3\pi}{2}$

$$\theta = \arctan\frac{1}{\sqrt{3}} + \pi = \frac{7\pi}{6}$$

$$\cos\varphi = \frac{z}{\rho} = \frac{\frac{5\sqrt{2}}{2}}{5} = \frac{\sqrt{2}}{2}, \qquad 0 \le \varphi \le \pi$$

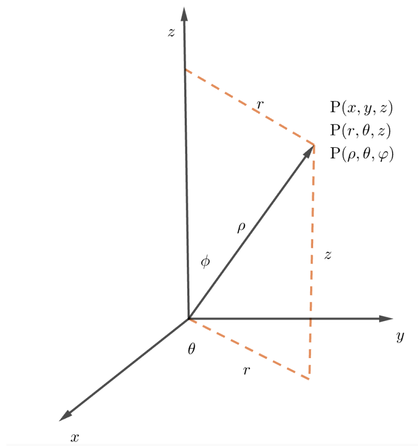
$$\Rightarrow \varphi = \arccos\frac{z}{\rho} = \frac{\pi}{4},$$

Exercise 7. (Related to WebWork Q9)

Convert the following point from spherical to Cartesian (rectangular) coordinates:

$$(\rho, \theta, \phi) = \left(1, \frac{3\pi}{4}, \frac{5\pi}{3}\right).$$

Solution.



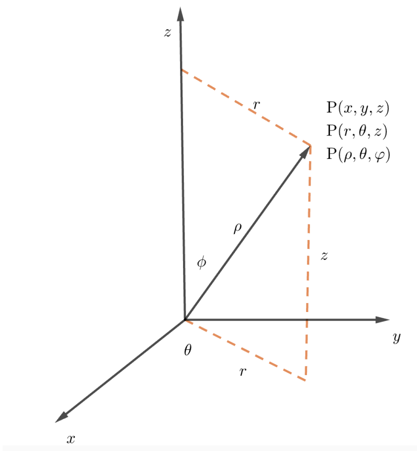
We are given that $\rho = 1, \theta = \frac{3\pi}{4}$, and $\phi = \frac{5\pi}{3}$. The relations between the spherical and rectangular coordinates imply

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta = 1 \sin\frac{5\pi}{3} \cos\frac{3\pi}{4} = 1 \cdot \frac{-\sqrt{3}}{2} \cdot \frac{-\sqrt{2}}{2} = \frac{\sqrt{6}}{4} \\
y &= \rho \sin\phi \sin\theta = 1 \sin\frac{5\pi}{3} \sin\frac{3\pi}{4} = 1 \cdot \frac{-\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{6}}{4} \\
z &= \rho \cos\phi = 1 \cos\frac{5\pi}{3} = 1 \cdot \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

Exercise 8. (Related to WebWork Q6)

What are the cylindrical coordinates of the point whose spherical coordinates are $(\rho, \theta, \phi) = \left(1, 1, \dfrac{5\pi}{6}\right)$?

Same as before, we can use right-triangle relationships to convert from one system to another.



From Spherical to Cylindrical	From Cylindrical to Spherical
$r = \rho \sin \phi$ $\theta = \theta$ $z = \rho \cos \phi$	$\rho = \sqrt{r^2 + z^2}$ $\theta = \theta$ $\phi = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$

Solution.

From Spherical to Cylindrical, we have

$$r = \rho \sin \phi$$
$$\theta = \theta$$
$$z = \rho \cos \phi$$

Thus

$$r = 1 \cdot \sin \frac{5\pi}{6} = 0.5$$
$$\theta = 1$$
$$z = 1 \cdot \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \approx -0.866025$$